

Black Holes and Spacetime Physics in String/M Theory

Miao Li

Institute of Theoretical Physics

Academia Sinica

Beijing 100080

and

Department of Physics

National Taiwan University

Taipei 106, Taiwan

`mli@phys.ntu.edu.tw`

In addition to briefly reviewing recent progress in studying black hole physics in string/M theory, we describe several robust features pertaining to spacetime physics that one can glean by studying quantum physics of black holes. In particular, we review 't Hooft's S-matrix ansatz which results in a noncommutative horizon. A recent construction of fuzzy AdS_2 is emphasized, this is a nice toy model for fuzzy black hole horizon. We demonstrate that this model captures some nonperturbative features of quantum gravity.

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1. Introduction

Quantum mechanics and Einstein's general relativity are both regarded as achievements of highest form in physics in twentieth century. Both have profoundly reformed our view of physical world, as well as have had many applications in explaining numerous observational facts. Yet, despite many heroic efforts, it has proven formidable feat to achieve to put both theories in a single theoretical framework. String theory has long been held by many as the most promising candidate for such a framework. Recent progress in unifying different string models into a single theory, the M theory, and in using string theory to understand certain aspects of quantum black holes, has reinforced this optimistic belief. String/M theory can not claim that its goal of unifying quantum mechanics and gravity has been achieved, mainly due to the lack of a fundamental formulation capable of describing all different situations even in principle.

To formulate a theory unifying quantum mechanics and general relativity, one feels that in the end the new theory must incorporate fundamental features of both theories. Quantum mechanics is a general kinematic theory supposed to be valid no matter what forces are involved, or what the detailed dynamics is. On the other hand, general relativity was a new invention of Einstein to reformulate gravity theory. It is dynamic by nature. However this is a very special dynamics, since it deeply involves spacetime which is taken as a fundamental ingredient in any kinetic theory. Thus one can not simply apply quantum mechanics to general relativity by treating the latter as just another dynamic system. In other words, the new theory must be kinematic as well as dynamic at the same time. It is very likely that when one attempts to include other forces or interactions into this theory, these interactions cannot be arbitrary, and must be an inseparable part of the holistic theory. This is surely the spirit of Einstein when he pursued a unified theory and dreamed to derive quantum from geometry in his late life. String theory seems to have this quality: When strings or things such as D0-brane partons are quantized, one gets fluctuations of spacetime for free along with other matter particles.

Many questions concerning the fate of spacetime become much sharpened in black holes [1]. It was observed by Bekenstein that a Schwarzschild black hole seems to represent the maximal entropy state of all physical systems that can be contained in a spherical region, and this entropy is proportional to the area of its horizon [2]. A black hole is not really black at the quantum level [3], it emits Hawking radiation. It turns out that in the entropy formula both the Planck constant and the Newton constant play a role, indicating

that a proper understanding of the origin of the entropy requires a theory of quantum gravity. This necessity is enhanced by the riddle of information loss [4]. It was believed by some that in the process of formation and evaporation of a black hole, quantum gravity effects play a minor role, so that the pure quantum state will evolve into a mixed state, and unitarity is lost. This is at odds with very general considerations of energy conservation [5], and it seems hard to build a theory with violation of unitarity at the Planck scale only. Among others, 't Hooft and Susskind have argued that infalling particles as well as Hawking particles of finite energy are extremely boosted to a stationary observer near the horizon [6,7], so that strong gravitational interaction among trans-Planckian particles is essentially involved. Based on this and the assumption that unitarity is preserved all the time, 't Hooft postulated his S-matrix ansatz [6], and noncommutativity of spacetime first appeared. Susskind advocated that some basic properties of string theory seem to fit the unconventional physics near a black hole. In particular, 't Hooft's reasoning leads to a drastic truncation of the Hilbert space, since for instance two high energy particles with opposite momenta within a fixed impact parameter is indistinguishable from a small black hole. This is one of reasons to propose the holographic principle [8]. This principle has gained much popularity recently through the celebrated Maldacena conjecture [9].

String theory has been successful in explaining the origin of entropy for various extremal and near extremal black holes, starting with the work [10]. This lends much support to the belief that string theory not only overcomes the perturbative divergences but also encodes correctly some nonperturbative features of spacetime physics. All constructions of extremal black holes involve D-brane configurations [11]. The D-brane theory is always weakly coupled when a reliable calculation can be done. In such a situation, spacetime curvature is large and therefore gravitational interactions are strong. Nevertheless one can still trust Bekenstein-Hawking formula since it is protected by supersymmetry. In some occasions, one still can trust D-brane theory even in its strong coupling limit, due to some nonrenormalization theorems. The black hole becomes macroscopic thus classic geometry emerges. Maldacena conjecture may be regarded, in somewhat technical view, as embodiment of infinitely many nonrenormalization theorems.

How to treat a Schwarzschild black hole in string/M theory still remains a big open problem, despite much work done in the context of matrix theory [12,13]. Physics can in principle be studied in matrix theory, however geometry as well as other related physical quantities are difficult to be recovered, just as in the D-brane approach to extremal black holes. It appears that new insights are needed in order to build a more fundamental and

transparent picture. We suspect that these insights will have much to do with quantum geometries which already showed up in 't Hooft's work, and with further understanding of holographic principle. Some recent proposals on fuzzy spheres and fuzzy anti-de Sitter spaces will provide a good starting point for pursuit in this direction [14,15]. We have every reason to believe that much progress will be made in near future.

In the next section, we start with a brief account of thermodynamic properties of a Schwarzschild black hole and an extremal black hole, with an emphasis on the unusual physics viewed by a stationary observer. Sect.3 reviews a calculation of extremal black holes in string theory. We will be brief again, since there exist several reviews on this subject [16]. We will discuss some work of 't Hooft in sect.4 with an emphasis on the quantum geometry arising from the S-matrix ansatz. We will also show in sect.5 that spacetime uncertainty in string theory has a similarity to the noncommutativity of 't Hooft. Schwarzschild black holes in matrix theory will be discussed in sect.6, here we will see that geometry is hard to study, this perhaps is the generic problem in all holographic theories. Finally we present some recent progress in understanding physics in fuzzy AdS_2 in sect.7.

We hope that the present short review article will help to call more attention to several very interesting subjects presented here, in addition to serving as a concise guide to more detailed original works.

Most of time we will use the natural unit in which $c = \hbar = 1$. The Newton constant G is not set to one, to emphasize one of the important length scales when the interactions are switched on. Without warning we sometimes reinstall \hbar to show the quantum origin of some effects.

2. Thermodynamics and Other Properties of Black Holes

Given suitable initial boundary conditions, it can be proven that formation of a black hole with event horizon is inevitable. For ordinary matter with interactions (or equation of state) dictated by known forces, it is well-known that a black hole forms of a burnt-out star of a few solar masses. It is possible that much smaller black holes can form of nonordinary matter such as scalar excitations in an early universe. It is also possible to form a microscopic black hole in a violent collision of extremely energetic particles.

The most studied, yet the most mysterious black hole from string theory viewpoint is the Schwarzschild black hole of mass M . Its metric

$$ds^2 = -(1 - (\frac{r_0}{r})^{D-3})dt^2 + (1 - (\frac{r_0}{r})^{D-3})^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \quad (2.1)$$

solves the D dimensional Einstein equations in vacuum, where $d\Omega_{D-2}^2$ is the metric on a unit round $D - 2$ dimensional sphere. At $r = r_0$, g_{00} degenerates, thus the red-shift factor becomes infinity viewed by an outside observer sitting at $r > r_0$. We will focus on the case $D = 4$ in this section. $r_0 = 2GM$ in this case, and we always set $c = 1$. Define

$$\rho = r + r_0 \ln(\frac{r}{r_0} - 1) \quad (2.2)$$

for $r > r_0$, the metric reads

$$ds^2 = -(1 - \frac{r_0}{r})dX^+dX^- + r^2 d\Omega_2^2, \quad (2.3)$$

where $X^\pm = t \pm \rho$. The first part of the metric vanishes at the horizon.

The Hawking temperature is given by

$$T_H = \frac{1}{4\pi} \partial_r g_{00}(r = r_0) = \frac{1}{4\pi r_0}, \quad (2.4)$$

here we have set both the Boltzmann constant $k = 1$ as well as the Planck constant $\hbar = 1$. Thus the temperature always has an energy unit. It is easy to see from (2.4) that the Hawking temperature is proportional to \hbar , so the origin of Hawking temperature is quantum mechanical. If the gravitational barrier is negligible for Hawking radiation, we see that the typical wave-length of Hawking radiation is the order of r_0 . This fact alone already indicates that a black hole is a unusual statistical system. Note that the Hawking temperature is the one viewed by a distant observer. The local temperature for an observer located at r is easily obtained by the red-shift formula

$$T(r) = T_H g_{00}(r)^{-1/2}. \quad (2.5)$$

For r close to r_0 , we have $T(r) = 1/(2\pi d(r))$, where $d(r) \sim 2\sqrt{r_0(r - r_0)}$ is the proper distance from the horizon. These formulas becomes more and more accurate when r_0 gets larger and larger. In fact, $T = 1/(2\pi d)$ is the formula for the Unruh temperature in the Rindler space

$$ds^2 = -e^{y^+ + y^-} dy^+ dy^-, \quad (2.6)$$

where $\exp y^\pm = X^\pm$ and X^\pm are the flat space light-cone coordinates. The proper distance at $\rho = \frac{1}{2}(y^+ + y^-)$ from the horizon is e^ρ and the Unruh temperature is $1/(2\pi)e^{-\rho}$. The fact that the geometry of a large black hole is close to the Rindler space makes it clear that study of quantum physics near a black hole will also bring about insights for physics in the flat spacetime.

The entropy of the black hole can be deduced from the first law of thermodynamics, assuming that indeed the black hole can be treated as a thermodynamic system. In order to see the quantum nature of the entropy, we reintroduce \hbar in T . Substitute $E = M = r_0/(2G)$ and $T = \hbar/(4\pi r_0)$ into $dE = TdS$, we find $dS = dA/(4G\hbar)$ where $A = 4\pi r_0^2$ is the area of the horizon. Integrating this formula we have

$$S = \frac{A}{4G\hbar} = \frac{A}{4l_p^2}, \quad (2.7)$$

we ignored an additive constant term which is presumably microscopic and nonuniversal. Note that S diverges in the limit $\hbar \rightarrow 0$, agreeing with the fact that a black hole is black in the classical limit. We deduced the entropy using the first law of thermodynamics, since it is the shortest derivation we can imagine. The original argument leading to (2.7) does not require thermodynamics at all. In fact, Bekenstein was motivated by the analogy of growth of the total horizon area in many dynamic processes to identify A with S [2]. The second law of thermodynamics is always valid.

The appearance of the Planck length l_p in the entropy formula suggests strongly to many people that the origin of this entropy must lie in quantum gravity, since l_p is the length scale at which quantum gravity effects become visible. The innocent looking of our derivation of (2.7) may tempt people to suspect that the area formula may be derived within quantum field theory in a fixed curved background, since Hawking radiation itself was derived in this way. The fact that this expectation is wrong can be seen from the following simple argument. Consider for instance a massless particle outside the horizon. By the usual statistical mechanics, the local entropy density is $aT^3(r)$. The integral

$$a \int T^3(r) g_{00}^{-1/2} r^2 dr = aT_H^3 \int r^2 g_{00}^{-2} dr \quad (2.8)$$

diverges linearly near $r = r_0$. It also diverges for large r . The latter divergence can be viewed from contribution of the bulk matter surrounding the black hole, and can be separated from the contribution near the horizon by a more careful analysis. The former divergence is a genuine UV property. It can be formally cut-off by introducing a cut-off on

the proper distance $d(r) \sim l_p$. Thus one would obtain an entropy formula similar to (2.7). This formal cut-off calls for a quantum gravity interpretation, and is termed sometimes as the brick-wall model.

A physical cut-off would typically violate Lorentz invariance, so the brick-wall model can hardly be taken literally. One may take the necessity of introducing a cut-off as implying that beyond the Planck temperature, the number of degrees of freedom becomes smaller and smaller. An alternative explanation seems even more attractive: Spacetime is fundamentally noncommutative, so when one probes the high energy regime, one reaches back to the long distance physics, thus, an integral such (2.8) is not well defined when both $1/T$ and $d(r)$ become small. It must be replaced by something else. We will see that indeed there is a Lorentz invariant quantum geometry which potentially offers such a formula.

Another puzzle concerns the origin of the Hawking radiation. The life time of the black hole can be obtained by integrating $dM/dt \sim -T_H^4 A \sim -1/r^2$. We have $t \sim r_0^3/l_p^2$. Suppose the Hawking radiation originate from a place a proper distance d away from the horizon. The time for a S-wave to propagate out is given by

$$t = \int g_{00}^{-1} dr \sim r_0 \ln \frac{r_0}{r - r_0} \sim 2r_0 \ln \frac{r_0}{d}. \quad (2.9)$$

If the brick wall is really there, it forms a perfect reflecting mirror, so for the majority of radiation to be able to escape it must be placed at d such that the above time is comparable to the life time of the black hole. We deduce from this

$$d \sim r_0 \exp\left(-\frac{r_0^2}{2l_p^2}\right), \quad (2.10)$$

this is an absurdly small distance compared to the Planck length for a macroscopic black hole.

An extremal or a near extremal black hole has been of central interest in the past few years in string theory. The simplest one comes from the Reissner-Nordström black hole. The 4D metric is

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_2^2 \quad (2.11)$$

with

$$f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} = \frac{1}{r^2}(r - r_+)(r - r_-), \quad (2.12)$$

where $r_{\pm} = GM \pm \sqrt{(GM)^2 - Q^2}$ are the radii of the outer and inner horizons respectively. Q is the electric charged carried by the black hole. The inequality $GM \geq |Q|$ must hold to avoid naked singularity. When the equality holds, the black hole is extremal. The time and radial coordinates exchange their roles when one crosses the outer horizon. Their roles are exchanged one more time when the inner horizon is crossed.

The Hawking temperature is obtained using formula (2.4), it is

$$T_H = \frac{1}{4\pi r_+^2}(r_+ - r_-). \quad (2.13)$$

It becomes zero for the extremal black hole. Thus it appears that the nonextremal charged black hole continues Hawking radiation until it rests as an extremal black hole. To obtain the entropy of a charged black hole, we use

$$dM = T_H dS + V dQ, \quad (2.14)$$

where V is the static potential at the outer horizon and assumes the value $V = Q/r_+$. With this formula and the Hawking temperature (2.13), (2.14) can be seen to be integrable and to yield

$$S = \frac{\pi r_+^2}{l_p^2} = \frac{A}{4l_p^2}. \quad (2.15)$$

Again the area law holds.

It appears that an extremal black hole is somewhat strange in that it has a large degeneracy, despite the fact that its Hawking temperature vanishes. Also, the brick-wall model seems to work again. But when one holds fixed the proper distance the brick wall from the outer horizon, the coordinate distance $r_* - r_+$ goes to zero as $1 - r_-/r_+$. An anti-de Sitter space is obtained from the extremal black hole by taking a scaling limit, see sect.7. Thus, the charged black hole is valuable for testing any idea implementing Planck scale physics.

3. Extremal Black Holes in String Theory

String theory distinguishes itself from other approaches to quantum gravity by postulating the existence of closed as well as open strings. The discovery of dualities makes it clear that for consistency other extended objects must be included too. The most important such objects are D-branes [11]. These branes carry charges with respect to

Ramond-Ramond antisymmetric tensor fields. Depending on which dual description one uses, strings can also be regarded as D-branes. D-branes are more fundamental in that their description contains open string excitations. These excitations are important in the counting of the number of states for various extremal black holes.

The single most important fact making string theory different in constructing black hole solutions is the existence of the dilaton field. This is a massless scalar field when the ground state has enough supersymmetry. Its expectation value determines the strength of string interactions. There are Schwarzschild black hole solutions in string theory. These are the same as in Einstein theory since in a vacuum the dilaton is a constant. However, various abelian gauge fields are coupled to the dilaton in one way or another, charged black holes are rather different from, say, the standard Reissner-Nordström black hole [17]. Thus, to obtain an extremal black hole with a macroscopic horizon, it is necessary to switch on several different charges, and the minimal number of charges is 3. The first such black hole is obtained in 5 dimensions [10]. Four charged black holes can be constructed in 4 dimensions.

We discuss only the 3 charged black hole in 5 dimensions, following [18]. Starting with IIB theory and compactifying it on T^5 , there is an abelian gauge field $C_{a\mu}^{(2)}$ resulting from $C^{(2)}$, the rank 2 R-R field. A wrapped D-string along X^a carries its electric charge. Another abelian gauge field, $C_{1,\dots,5,\mu}^{(6)}$ results from the dual of $C^{(2)}$. A D5-brane wrapped around T^5 carries its electric charge. Two unbroken supersymmetry conditions $\epsilon = \gamma^{0a}\tilde{\epsilon}$ and $\epsilon = \gamma^{01\dots 5}\tilde{\epsilon}$ are compatible if a is one of $1, \dots, 5$. That is, the bound state of N_5 D5-branes wrapped around T^5 and N_1 D-strings wrapped around a circle of T^5 is a BPS state. The residual SUSY is 1/4 of the number of original supersymmetry. Take $a = 1$.

We need one more charge to construct a black hole. This is achieved by adding momentum modes along X^1 , namely along the D-string direction. This introduces a further constraint on unbroken SUSY $\epsilon = \gamma^{01}\epsilon$, $\tilde{\epsilon} = \gamma^{01}\tilde{\epsilon}$. This means that both ϵ and $\tilde{\epsilon}$ are positive eigenstates of γ^{01} . Combined with the D-string constraint, $\epsilon = \tilde{\epsilon}$. Thus there are 8 unbroken super-charges. Finally the D5-brane constraint eliminates half of them. The BPS black hole preserves 4 super-charges.

To see that this is a black hole, we need the metric:

$$\begin{aligned}
ds^2 &= (H_1 H_5)^{-1/2} (-dt^2 + dX_1^2 + (H_p - 1)(dt - dX_1)^2) + H_1^{1/2} H_5^{-1/2} (dX_2^2 + \dots + dX_5^2) \\
&\quad + (H_1 H_5)^{1/2} (dr^2 + r^2 d\Omega_3^2), \\
e^{2\phi} &= g^2 H_1 H_5^{-1},
\end{aligned} \tag{3.1}$$

where H_i are harmonic functions in 5 dimensions, $H_i = 1 + r_i^2/r^2$, where the parameter r_i^2 is proportional to the corresponding charge. And the R-R fields

$$C_{01}^{(2)} = \frac{1}{2}(H_1^{-1} - 1), \quad F_{ijk} = \frac{1}{2}\epsilon_{ijkl}\partial_l H_5, \quad (3.2)$$

where i, j, k, l are indices tangent to the 4 open spatial dimensions. Let $(2\pi)^4 V$ denote the volume of T^4 orthogonal to the D-strings, and R_1 the radius of X^1 . It is easy to see that

$$r_1^2 = \frac{gN_1}{V}, \quad r_5^2 = gN_5, \quad r_p^2 = \frac{g^2 N_p}{R_1^2 V}, \quad (3.3)$$

where the momentum along X^1 is N_p/R_1 . We have set $2\pi\alpha' = 1$. For fixed V and R_1 , we see that all sizes r_i become macroscopically large when $gN_1 \gg 1$, $gN_5 \gg 1$, and $g^2 N_p \gg 1$. We call this region of the parameters the black hole phase.

When reduced to 5D, the Einstein metric reads

$$ds^2 = -(H_1 H_5 H_p)^{-2/3} dt^2 + (H_1 H_5 H_p)^{1/3} (dr^2 + r^2 d\Omega_3^2). \quad (3.4)$$

From the component G_{00} we see that $r = 0$ is the horizon, since the red-shift factor becomes infinity at this point. The Bekenstein entropy is easy to calculate, either by using the 8 dimensional horizon if the hole is treated as living in 10 dimensions, or by using the 3 dimensional horizon when it is treated as living in 5 dimensions. It is relatively simpler to use the 5D metric. The horizon area is given by $2\pi^2[r^2(H_1 H_5 H_p)^{1/3}]^{3/2}$ when the limit $r \rightarrow 0$ is taken. Thus $A_3 = 2\pi^2 r_1 r_5 r_p$, and $r = 0$ is not a point. The 5D Newton constant is $G_5 = g^2/(4V R_1)$, so the entropy is

$$S = \frac{A_3}{4G_5} = 2\pi\sqrt{N_1 N_5 N_p}, \quad (3.5)$$

a nice formula.

It can be shown that all 5 dimensional black holes preserving 1/8 of supersymmetry can be rotated into above black hole using U-duality, here the U-duality group is E_6 . If one can count the entropy microscopically for one of them, then others must have a microscopic origin too on the count of U-duality. For instance, a 5D black hole in IIA theory is obtained by performing T-duality along X^1 . The hole is built with D4-branes, D0-branes bound to them, and string winding modes around the dual of X^1 . Now this has a simple M theory interpretation, the D4-branes get interpreted as fivebranes wrapped around the M circle, winding strings get interpreted as membranes wrapped around the

M circle, and D0-branes are M momentum modes. Thus, the hole is built using fivebranes intersecting membranes along a circle with momentum modes running along this circle.

Come back to the IIB 5D black hole. The simplest account of the microscopic picture goes as follows. The D-strings are bound to D5-branes, and they live on the Higgs branch in the weak string coupling limit, thus can oscillates only in the 4 directions along D5-branes. If the size of V is much smaller than R_1 , the oscillation is effectively described by a 1+1 conformal field theory. The fluctuations correspond to wiggling of the open strings stretched between D5-branes and D-strings, thus there are $4N_1N_5$ such bosons. Due to supersymmetry, there are also the same number of fermions. The theory is therefore a conformal field theory with central charge $6N_1N_5$. Since in a CFT a fluctuation is either left-moving or right-moving, and we restrict our attention to BPS states, there are only right-moving modes which contribute to the total momentum N_p/R_1 . Thus, N_p is the oscillator number. We are therefore interested in the coefficient of q^{N_p} in the expansion of the following partition function

$$Z = \left(\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} \right)^{4N_1N_5}, \quad (3.6)$$

and it is given, after a saddle point calculation, by $\exp(2\pi\sqrt{N_1N_2N_p})$, that is, the entropy agrees exactly with (3.5).

There is subtlety involved in the above calculation, making it invalid for large N_1 and N_5 . A cure of this problem is provided by the fractionation mechanism, whose details we will not run into here [19].

The Hawking temperature of the 5D extremal black hole is zero, so it does not Hawking radiate. To obtain a nonextremal black hole, we need to add some anti-charges to the hole. The metric (3.1) is modified by a nonextremal factor $f(r) = 1 - r_0^2/r^2$ to the part dt^2 , and f^{-1} to the part dr^2 . The location of the horizon is shifted to r_0 . The simplest way to obtain this metric is by adding momentum modes moving in the opposite direction along X^1 compared to the existing modes. Now the Hawking temperature is proportional to the square root of the number of the added modes. Assuming that the Hawking radiation comes mainly from combination of the left-moving modes and the right-moving modes, a careful calculation by Das and Mathur [20] shows that the perturbative string calculation reproduces exactly the correct Hawking's black-body formula. It is even more striking that a further perturbative string calculation reproduces the correct grey-body factor [21], which is just the energy-dependent absorption cross section.

All the above discussions can be extended to the 4D extremal and near extremal black holes [22], and to charged, spinning black holes [23].

Now matter how nice the results one can obtain using the D-brane technology, one can not help but feel that there is something crucial missing. One does not have a geometric picture at all, since the D-brane account is more or less holographic. With the advent of Maldacena conjecture, understanding emergence of geometry has become more urgent. In this regard, it is perhaps useful to go back where one started in first place.

4. 't Hooft's S-matrix Ansatz

Having seen impressive progress made in string theory explaining some of the quantum properties of near extremal black holes, we come back to issues concerning the Schwarzschild black holes. The main reason for doing this is not only that these black holes are still poorly understood in string theory, but also that understanding these black holes will bring in new insights and perhaps it is about the right time to look back at some points having great potential in near future.

One interesting approach is 't Hooft's S-matrix ansatz [6]. 't Hooft postulates that quantum mechanics is always valid during the whole process of formation and evaporation of a black hole, and that the trans-Planckian regime can not be ignored when one considers the effects of incoming particles on the Hawking radiation. As we already pointed out in sect.2, very close to the horizon, due to the enormous red-shift factor, an infalling particle as well as an outgoing particle with finite energy measured at infinity are boosted to very high energies. Whether they are massive or massless is immaterial. To see the mutual effect between an incoming particle and an outgoing particle, let us examine the gravitational field produced by a shock-wave in the flat spacetime first.

Consider an massless particle moving in the direction x^3 . For a left-moving particle, its wave function is a function of $x^+ = t + x^3$. The stress tensor thus has a nonvanishing component T_{++} . A shock-wave is defined by the characteristic that the energy is concentrated at a definite value, that is, $T_{++} \sim \delta(x^+)$, here for convenience we choose the concentrating point to be $x^+ = 0$. Physically, the massless particle passes the point $x^3 = 0$ at $t = 0$. Since spacetime is 4 dimensional, we also need to specify the location of the particle in the transverse space (x^1, x^2) . Again for simplicity we take the location to be

the origin. Due to the rotational invariance in the transverse space, it is easy to convince oneself that the metric produced by the shock-wave assumes the form

$$ds^2 = -dx^+ dx^- + \delta(x^+) h(r) (dx^+)^2 + dx^i dx^i, \quad (4.1)$$

where x^i runs through x^1, x^2 , and $r^2 = (x^1)^2 + (x^2)^2$. The metric is flat away from $x^+ = 0$. For $T_{++} = \delta(x^+) T(x^i)$, the Einstein equation reduces to

$$\Delta h(x^i) = 8\pi G T(x^i). \quad (4.2)$$

Consider the special case $T(x^i) \sim \delta^2(x)$. Since by the definition

$$p_+ = \int T_{++} dx^+ d^2x,$$

we have $T_{++} = p_+ \delta(x^+) \delta^2(x)$. And the solution to the Einstein equation is

$$h = 4G p_+ \ln r. \quad (4.3)$$

(4.3) together with (4.1) is called Aichelburg-Sexl metric [24].

The effect of the metric (4.1) on a right-moving particle can be seen easily for a massless particle. The classical trajectory in a flat spacetime is specified by $dx^- = 0$. However, in a background with a nonvanishing component g_{++} , the trajectory is changed to $dx^- = g_{++} dx^+$. For g_{++} of the form (4.1), the trajectory is $x^- = \text{const}$ on the both sides separated by $x^+ = 0$. There is a jump when the right-moving particle crosses $x^+ = 0$, the jump is

$$\Delta x^- = 4l_p^2 p_+ \ln r, \quad (4.4)$$

if the location of the right-moving particle on the transverse space is r , we have also replaced G by l_p^2 .

Apparently, the result (4.4) is a purely classical result, despite the way we write it. In fact $l_p^2 p_+$ can be roughly viewed as the measure of the gravitational size, since it is roughly the Schwarzschild black hole radius if there is a right-moving particle with the same energy colliding with the left-moving one and forming a black hole. The validity of (4.4) requires r to be larger than a certain scale. Lacking a detailed quantum gravity theory, we can only guess that this scale can be either l_p or $l_p^2 p_+$ itself.

We generalize the above analysis to the geometry near a black hole horizon. We introduced tortoise coordinate (2.2) in sect.2, although the interesting part of the metric

(2.3) is conformal to the flat metric, it is inconvenient to work with ρ since the horizon is located at $\rho = -\infty$. The more convenient coordinate system is Kruskal coordinates x^\pm , the relations between the two systems are simply

$$x^\pm = \exp(\pm \frac{X^\pm}{2r_0}). \quad (4.5)$$

The past horizon corresponds to $X^+ = -\infty$, or $x^+ = 0$, the future horizon corresponds to $X^- = \infty$ or $x^- = 0$. An incoming particle moves with fixed x^+ and with decreasing x^- , and an outgoing particle moves with a fixed x^- and with increasing x^+ . The Schwarzschild metric is

$$ds^2 = \frac{4r_0^3}{r} e^{-r/r_0} dx^+ dx^- + r^2 d\Omega_2^2. \quad (4.6)$$

Denote $2A$ the prefactor in the front of $dx^+ dx^-$ in the above formula, the metric produced by an incoming shock-wave takes the general form

$$ds^2 = 2A dx^+ dx^- - 2\delta(x^+) A F(dx^+)^2 + r^2 d\Omega_2^2, \quad (4.7)$$

where F is a function of x^- and angular variables of the sphere S^2 . 't Hooft calculated the Ricci tensor of the above metric, and found that the only nonvanishing component is R_{++} . Let Δ be the Laplacian on the unit sphere, Einstein equation reads simply

$$\Delta F - F = 8\pi \frac{G}{r_0^2} p_+ \delta(\Omega), \quad (4.8)$$

where $\delta(\Omega)$ is the delta function on the unit sphere.

If the incoming particle locates at the north pole of the two sphere, the solution to (4.8) is

$$F(\theta) = \frac{4G}{r_0^2} p_+ f(\theta) = -\frac{4G}{r_0^2} p_+ \sum_l \frac{l + \frac{1}{2}}{l(l+1) + 1} P_l(\cos \theta). \quad (4.9)$$

The effect of the incoming shock wave on an outgoing particle is therefore a shift in x^-

$$\Delta x^- = \frac{4G}{r_0^2} p_+ f(\theta). \quad (4.10)$$

This shift is a constant. However, in terms of X^- , the shift is proportional to $\exp(X^-/(2r_0))$. For an outgoing particle originating sufficiently close to the future horizon, where $X^- = \infty$, this shift is enormous.

Now we can construct a piece of the S-matrix in the background of a black hole. A particle or a state of multiple-particles coming from infinity near the horizon ($r - r_0 \sim r_0$)

is described by S_{in} , this part is governed more or less by the known theory. Similarly, S_{out} describes how outgoing particles leave from the region $r - r_0 \sim r_0$. A nontrivial piece, denoted by S_{hor} , describes the effect of incoming particles on the outgoing particle very close to the horizon. Thus the S-matrix in a fixed black hole background is $S = S_{out}S_{hor}S_{in}$. This splitting is approximate only. We are mostly interested in S_{hor} .

For outgoing particles, the phase is shifted by an amount $\exp(-ip_- \delta x^-)$. Thus

$$\psi_{out} = e^{-i \int d\Omega P_-(\Omega) \delta x^-(\Omega)} \psi_{in}, \quad (4.11)$$

where $P_-(\Omega)$ is the outgoing momentum density operator. Using (4.10), the above is rewritten

$$\psi_{out} = e^{-i \int d\Omega d\Omega' P_-(\Omega) F(\Omega, \Omega') P_+(\Omega')} \psi_{in}, \quad (4.12)$$

where $P_+(\Omega')$ is the incoming momentum density operator. The near hole S-matrix reads simply

$$S_{hor} = \mathcal{N} \exp \left(-i \int d\Omega d\Omega' P_-(\Omega) F(\Omega, \Omega') P_+(\Omega') \right). \quad (4.13)$$

This is the main result of 't Hooft. The above S-matrix is not satisfactory, since it still assumes that the transverse part of the horizon S^2 is a continuous surface, and the function F suffers a logarithmic divergence when two points on S^2 get close. This can be cured by generalizing the shock-wave of carrying only longitudinal momentum to one also carrying some transverse momentum. One is lead to some kind of fuzzy sphere. This is quite similar to the recently discovered fuzzy spheres in the AdS/CFT correspondence [15].

The S-matrix ansatz elevates the classical result (4.10) to a quantum mechanic one. One may go one step further to claim that (4.10) implies a commutator between x^+ and x^- :

$$[x^+(\Omega), x^-(\Omega')] = \frac{4G}{r_0^2} f(\Omega, \Omega') [x^+, p_+] = \frac{4iG\hbar}{r_0^2} f(\Omega, \Omega'), \quad (4.14)$$

thus the Planck length $l_p^2 = G\hbar$ automatically appears after a simple incorporation of quantum mechanics. We will see that the fuzzy AdS_2 model exhibits a very similar commutation relation between the light-cone coordinates, although the initial motivation for proposing this model is quite independent of the shock-wave argument [15].

A physically nontrivial consequence of the S-matrix (4.13) is teleology. An observer who sees an outgoing particle would deduce that an incoming particle is affected: If one trace back along the trajectory of the outgoing particle, it also induces an enormous shift on an incoming particle. Thus an infalling observer would conclude that what he sees is correlated with what a distant observer sees. This leads to the conclusion that operators with spacelike separation do not commute. A recent discussion on this phenomenon in a noncommutative field theory and string theory can be found in [25].

5. Spacetime Uncertainty in String/M Theory

The first string revolution brought about powerful perturbative tools in studying string theory, and a classification of perturbative string vacua. Although a number of interesting things concerning the nature of space were discovered, such as T-duality and mirror symmetry, the nature of space and time was largely obscure. The only exception was the pioneering work of Yoneya [26], [27] on a spacetime uncertainty relation. This proposal had been ignored by the community until the second string revolution. The first revival of interest in this relation was a check in D-brane dynamics [28]. This relation is rather stringy, and its M theory generalization was proposed in [29]. Further elaborations on this subject are presented in a beautiful review [30].

In the perturbative formulation of string theory, conformal invariance on the world-sheet plays a fundamental role. It is therefore important to extract from conformal invariance some physical property which may survive interactions. Here we follow the original approach of [26] to derive a spacetime uncertainty relation. Consider a parallelogram on the string world-sheet and the Polyakov amplitude for the mapping from the world-sheet to a region of the target space-time. Let the lengths of the two orthogonal sides in the world-sheet parallelogram be a, b in the conformal gauge where $\dot{x} \cdot x' = 0$, $\dot{x}^2 + x'^2 = 0$ and the corresponding physical space-time length be A, B , respectively. Then apart from the power behaved pre-factor, the amplitude is proportional to

$$\exp\left[-\frac{1}{\ell_s^2}\left(\frac{A^2}{\Gamma} + \frac{B^2}{\Gamma^*}\right)\right] \quad (5.1)$$

where

$$\Gamma \equiv \frac{a}{b}, \quad \Gamma^* \equiv \frac{b}{a}, \quad (\Gamma\Gamma^* = 1). \quad (5.2)$$

Due to the conformal invariance, the amplitude depends on the Riemann sheet parameters only through the ratio Γ or Γ^* , which are called the *extremal length* and the conjugate extremal length, respectively. Clearly, the relation $\Gamma\Gamma^* = 1$ leads to the uncertainty relation

$$\Delta T \Delta X \sim \ell_s^2 \sim \alpha' \quad (5.3)$$

taking the A direction to be time-like $\Delta T \sim \langle A \rangle$ and hence the B direction to be space-like $\Delta X \sim \langle B \rangle$. The obvious relation $\Gamma\Gamma^* = 1$ is the origin of the familiar modular invariance of the torus amplitudes in string theory. The extremal length is the most fundamental

moduli parameter characterizing conformal invariants, in general. Since arbitrary amplitudes can be constructed by pasting together many parallelograms, any string amplitudes satisfy the above reciprocal relation qualitatively. Although this form looks too simple as the characterization of the conformal invariance, it has a virtue that its validity is very general, as we will explain shortly, and does not use the concepts which depend intrinsically on perturbation theory. Our proposal is to use this relation as one of possible guiding principles towards nonperturbative reformulation of string theory and M-theory.

The uncertainty relation (5.3) is consistent with an elementary property of strings that the energy of a string is roughly proportional to its space-time length.

$$\Delta E \sim \frac{\hbar}{\alpha'} \Delta X_l. \quad (5.4)$$

with X_l being the length of a string measured along its longitudinal direction. Then the ordinary time-energy uncertainty relation $\Delta T \Delta E \geq \hbar$ leads to (5.3). It is important here to discriminate the length scales in the longitudinal and transverse directions with respect to the string. As is well known, transverse length scale grows logarithmically with energy used to probe the strings. This explains the linearly rising Regge trajectory for the Regge-pole behavior in high-energy peripheral scattering. The dual role of the time and the longitudinal spatial lengths is a natural space-time expression of the original s - t duality. The particle exchange (Regge exchange in the old language) and the resonance behaviors correspond to the regimes, $\Delta X_l \rightarrow \infty$ and $\Delta T \rightarrow \infty$, respectively. Furthermore, the Regge behavior is consistent with the existence of graviton, since scattering amplitudes in general are expected to be roughly proportional to $\Delta X_l \sim \ell_s^2 / \Delta T \propto E$ which implies, by adopting the argument in [31], that the intercept $\alpha(0)$ of the leading Regge trajectory is 2, from the relation $E \sim E^{\alpha(t)-1}$.

On the other hand, in the high energy fixed-angle scatterings with large s - and t -channel momenta studied in detail in [32], we are trying to probe the region where both the time and the spatial scales are small. Clearly, such a region is incompatible with the space-time uncertainty relation. The exponential fall-off of the perturbative string amplitudes in this limit may be interpreted as a manifestation of this property. According to the space-time uncertainty relation, at least one of the two length scales, ΔX or ΔT , must be larger than the string scale ℓ_s . Therefore there is no degrees of freedom left in this regime. However, when one really tries to extract spacetime information from the fixed angle amplitudes, one finds that they are compatible with the relation (5.3),

as recently discussed in [30]. It is well known that any consistent theory of quantum gravity must indicate lessening of the degrees of freedom near the Planck scale where the quantum nature of gravity becomes important. We have seen that the space-time uncertainty relation can indeed be a natural mechanism for this. Qualitatively, it is also consistent with the known high-temperature behavior of the perturbative string amplitude, since the high-temperature limit is effectively equivalent to considering the limit $\Delta T \rightarrow 0$.

To check that the spacetime uncertainty relation hold in D-brane dynamics, we consider the simplest, perhaps most important case of D0-branes. Consider the scattering of two D0-branes of mass $1/g_s \ell_s$ with the impact parameter of order ΔX and the relative velocity v which is assumed to be much smaller than the light velocity. Then the characteristic interaction time ΔT is of order $\frac{\Delta X}{v}$. Since the impact parameter is of the same order as the longitudinal length of the open strings mediating the interaction of the D-particles, we can use the space-time uncertainty relation in the form

$$\Delta T \Delta X \sim \ell_s^2 \Rightarrow \frac{(\Delta X)^2}{v} \sim \ell_s^2$$

This gives the order of the magnitude for the minimum possible distances probed by the D-particle scatterings with velocity $v (\ll 1)$.

$$\Delta X \sim \sqrt{v} \ell_s. \tag{5.5}$$

To probe short spatial distances, we have to use D-particles with small velocity. However, the slower the velocity is and hence the longer the interaction time is, the larger is the spreading of the wave packet.

$$\Delta X_w \sim \Delta T \Delta_w v \sim \frac{g_s}{v} \ell_s, \tag{5.6}$$

since the ordinary time-energy uncertainty relation says that the uncertainty of the velocity is of order $\Delta_w v \sim g_s v^{-1/2}$ for the time interval of order $\Delta T \sim v^{-1/2} \ell_s$. Combining these two conditions, we see that the shortest spatial length is given by

$$\Delta X \sim g_s^{1/3} \ell_s \tag{5.7}$$

and the associated time scale is

$$\Delta T \sim g_s^{-1/3} \ell_s. \tag{5.8}$$

(5.7) is of course the 11 dimensional Planck length which is the characteristic length of M-theory which was first derived in the super YM context in [33]. As argued in [28], it is actually possible to probe shorter lengths than the Planck length if we consider a D-particle in the presence of many ($=N$) coincident D4-branes.

One can extract the M theory uncertainty relation from the stringy one valid for D0-branes. This derivation assumes the validity of matrix theory [34]. Note that a process involving individual D-partons necessarily smears over the longitudinal direction, thus the uncertainty in this direction $\Delta X_L = R$ is maximal. Relation (5.3) is rewritten as

$$\Delta X_T \Delta X_L \Delta T \geq l_p^3, \quad (5.9)$$

this relation refers only to the fundamental length scale in M theory, the Planck length, thus it is a natural candidate for the generalized uncertainty relation in M theory. We now argue that relation (5.9) is the correct relation for a process involving a boosted cluster. It is trivially true for threshold bound state, since it is just a boosted parton and according to Lorentz invariance ΔX_L contracts, while ΔT is dilated by a same factor. An object carrying the same amount of longitudinal momentum can be regarded as an excited state of the threshold bound state, therefore intuitively as a probe it can not probe a transverse distance shorter than a threshold bound state can do. Thus, relation (5.9) must also hold for such a probe. Note also that this relation is Lorentz invariant.

When the new relation (5.9) was proposed in [29], it was also checked that this relation is valid in the AdS/CFT correspondence [9], where conformal invariance on the M-branes is essentially employed. More recently, it was checked that this relation is compatible with the stringy exclusion principle, based on a remarkable dipole mechanism proposed in [14]. We hope that a precise mathematical framework properly accommodating (5.9) will offer a clue to a covariant formulation of matrix theory. Some attempts to constructing new brackets which may be relevant to the cubic uncertainty relation can be found in [35].

It remains to connect the more microscopic relations (5.3), (5.9) to the noncommutative black hole horizon we discussed in the previous section.

6. Schwarzschild Black Holes in Matrix Theory

Matrix theory promises us a nonperturbative definition of M theory in 11 dimensions, as well as toroidal compactification on a torus T^d with $d \leq 5$. If so, a Schwarzschild black hole must be in principle describable in matrix theory. Indeed many semi-quantitative

results were obtained in this framework, such as the scaling law between the mass of the black hole and its radius [12], [13]. Nevertheless matrix theory has the reputation of unwieldy, so by far it is impossible even to extract the standard Schwarzschild geometry.

The simplest situation is when the radius of the black hole, after boosted, matches the infrared cut-off size in the longitudinal direction in matrix theory. Naively, the radius of a black hole, like everything else in special relativity, contracts in the longitudinal direction with a boost: $r_0 \rightarrow e^{-\beta} r_0$, where β is the boost parameter. For the matrix theory to be effective, $r_0 \gg R$. The longitudinal momentum scales inversely with $e^{-\beta}$, so $P_- \sim r_0 M/R$. Since in matrix theory, all longitudinal momentum is carried by the partons, $P_- = N/R$, N is the number of partons, we have $N \sim r_0 M$. This number is the same magnitude of the black hole entropy, since in D dimensional spacetime, there are relations

$$S \sim \frac{r_0^{D-2}}{G}, \quad M \sim \frac{r_0^{D-3}}{G}, \quad (6.1)$$

where G is the D dimensional Newton constant. That the number of partons required matches roughly the entropy of the black hole strongly suggests that the partons are responsible for counting of microscopic states. Also note that for matrix theory to be able to accommodate the black hole, N is related to the minimal boost, thus N may be understood as the minimal number of partons required to describe the black hole microscopically.

It was later pointed out in [13] that the Lorentz contraction really occurs to the size of the black hole as seen by a distant observer, and that the actual size of the horizon is not changed, as suggested the purely geometric definition of the black hole horizon, which doesn't depend on which coordinates system one uses. Interpreted by a distant observer, who actually uses matrix theory to describe the black hole, the size of the black hole becomes larger for a probe near the horizon, and this enlargement is due to the pressure exerted by partons carrying longitudinal momentum. Before running into any details, we already see that it is going to be hard to study geometry in matrix theory, since as an input, the geometry is always fixed at spatial infinity, and in order to study geometry generated by sources, we need first of all define new geometric quantities in matrix theory.

There is a very simple derivation of the relation between the radius and the mass in matrix theory. As we said above, the longitudinal radius is not easy to see, however, the transverse radius is not distorted by boost, thus can be seen directly in matrix theory. When $D = 11$, consider the effective interaction between two partons in the leading order

$$V = c \frac{l_p^9}{R^3} \frac{(v_1 - v_2)^4}{r^7}, \quad (6.2)$$

where c is a numerical constant. The interaction is attractive, thus c is negative, if V is taken as a contribution to the effective two-body Hamiltonian.

Here are two crucial points in reconstructing the black hole data. First, every parton has a thermal wave length comparable to the size of the hole. This is a highly nontrivial assumption, as we know that this is far from being generic in a thermal system. Second, the hole may be regarded as a metastable system such that one can treat it as a stable bound state for all practical purposes. From the first assumption, one gets

$$v \sim \frac{R}{r_0}, \quad (6.3)$$

this is equal to the boost factor. From the second assumption, one has the virial theorem

$$\frac{1}{2} \frac{v^2}{R} \sim N \frac{l_p^9 v^4}{R^3 r_0^7}. \quad (6.4)$$

Combining these two equations, we derive

$$N \sim \frac{r_0^9}{l_p^9}, \quad (6.5)$$

that is, the number of partons is approximately the entropy of the black hole. This estimate is independent of the previous “fit the box” argument, thus we expect that by combining that argument with the above formula, we will get the relation between M and N . Alternatively, by substituting (6.5) into the matrix Hamiltonian, one gets

$$H = E_{LC} \sim \frac{1}{2} \frac{N}{R} v^2 \sim \frac{R r_0^7}{l_p^9}, \quad (6.6)$$

and the mass

$$M = \sqrt{P_- E_{LC}} \sim \frac{r_0^8}{l_p^9}. \quad (6.7)$$

This is precisely the mass formula for the hole. This derivation is independent of the boost argument we gave in the beginning of this section. It is difficult to justify the boost argument in the present context, since Lorentz boost properties are hard to study.

One way to probe the geometry near the lump of partons as a black hole is to study probes. The simplest probe is the D0-brane parton itself. This project was initiated in the third paper of [13]. In a boosted black hole geometry, the D0-brane action is the generalized Born-Infeld action. In order to probe the full geometry, one needs to calculate the interaction between the probe and the lump up to all orders in the double expansion in the velocity and the distance. This is hard to do. Thus so far it is fair to say that even the classical geometry of the matrix black hole has not been extracted.

The even more interesting problem is to study the quantum process of scattering a D0-brane parton against the lump, in order to extract the quantum geometry similar to the one proposed by 't Hooft.

7. Fuzzy $AdS_2 \times S^2$

7.1. Fuzzy $AdS_2 \times S^2$ from AdS/CFT Correspondence

Consider the near horizon limit of a 4 dimensional charged black hole in string theory [36]. For instance, by wrapping two sets of membranes and two sets of M5-branes in T^7 , one obtains a 4D charged, extremal black hole [38]. The brane configuration is as follows. Denote the coordinates of T^7 by x_i , $i = 1, \dots, 7$. A set of membranes are wrapped on (x_1, x_2) , another set are wrapped on (x_3, x_4) . A set of M5-branes are wrapped on $(x_1, x_3, x_5, x_6, x_7)$, the second set are wrapped on $(x_2, x_4, x_5, x_6, x_7)$. By setting all charges to be N , one finds the metric of $AdS_2 \times S^2$ for (x_0, x_8, x_9, x_{10}) :

$$ds^2 = l_p^2 \left(-\frac{r^2}{N^2} dt^2 + \frac{N^2}{r^2} dr^2 + N^2 d\Omega_2 \right), \quad (7.1)$$

$$F = -N d\Omega_{1+1} - N d\Omega_2,$$

where l_p is the 4 dimensional Planck length, $d\Omega_{1+1}$ and $d\Omega_2$ are the volume forms on AdS_2 and S^2 , respectively. The field F is just the linear combination of all anti-symmetric tensor fields involved. Note that here for simplicity, we consider the most symmetric case in which all the charges appearing in the harmonics $1 + Q_i l_p / r$ are just N which in turn is equal to the number of corresponding branes used to generate this potential. As a consequence, the tension of the branes compensates the volume of the complementary torus. This means that the size of each circle of T^7 is at the scale of the M theory Planck length.

The same space $AdS_2 \times S^2$ can also be obtained by taking the near horizon limit of the 4 dimensional extremal Reissner-Nordström solution.

In [15] we proposed that the S^2 part of the $AdS_2 \times S^2$ space is a fuzzy S^2 [37] defined by

$$[Y^i, Y^j] = i\epsilon^{ijk} Y^k, \quad (7.2)$$

where Y_i 's are the Cartesian coordinates of S^2 . (We use the unit system in which $l_p = 1$.) This algebra respects the classical $SO(3)$ invariance.

The commutation relations (7.2) are the same as the $SU(2)$ Lie algebra. For the $(2N+1)$ dimensional irreducible representation of $SU(2)$, the spectrum of Y_i is $\{-N, -(N-1), \dots, (N-1), N\}$. and its second Casimir is

$$\sum_{i=1}^3 (Y_i)^2 = N(N+1). \quad (7.3)$$

Since the radius of the S^2 is Nl_p (in the leading power of N), we should realize the Y_i 's as $N \times N$ matrices on this irreducible representation of $SU(2)$.

One evidence for this proposal is the following. For a fractional membrane wrapped on (x_1, x_3) or (x_2, x_4) , it is charged under the F field generated by a set of M5-branes. Denote the polar and azimuthal angles of S^2 by (θ, ϕ) . The stable trajectories of the membrane with the angular momentum M are discrete.

$$\cos \theta = \frac{M}{N}, \quad (7.4)$$

and they all have the same energy of $1/N$. It follows that, since M is conjugate to ϕ , $\cos \theta$ and ϕ do not commute with each other in the quantized theory. The resulting Poisson structure on S^2 is precisely that of the fuzzy sphere.

In [39], it was proposed that in $2+1$ dimensions the spacetime coordinates are quantized according to

$$[x, y] = \frac{i}{\cos^2 \mu} L, \quad (7.5)$$

where $\cos \mu$ is related to the mass of the particle, and L is the angular momentum on the 2 dimensional space. To complete the algebra we write down the usual relations

$$[L, x] = iy, \quad [L, y] = -ix. \quad (7.6)$$

This algebra is rotational invariant, and its $3+1$ dimensional generalization was given by Yang [40] much earlier. Note that this algebra (7.5) was proposed based on general grounds for a gravitational theory in $2+1$ dimensions, and we are content with the fact that it is actually a consequence of the algebra of the fuzzy S^2 (7.2) for massless particles ($\cos \mu = 1$), where Y_3 acts on Y_1 and Y_2 as the angular momentum.

In [15] we further proposed that the AdS_2 part is also quantized. Let X^{-1}, X^0, X^1 be the Cartesian coordinates of AdS_2 . The algebra of fuzzy AdS_2 is

$$\begin{aligned} [X^{-1}, X^0] &= -iX^1, \\ [X^0, X^1] &= iX^{-1}, \\ [X^1, X^{-1}] &= iX^0, \end{aligned} \quad (7.7)$$

which is obtained from S^2 by a ‘‘Wick rotation’’ of the time directions X^0, X^{-1} . The ‘‘radius’’ of AdS_2 is $R = Nl_p$, so that

$$\eta_{ij} X^i X^j = (X^{-1})^2 + (X^0)^2 - (X^1)^2 = R^2, \quad (7.8)$$

where $\eta = \text{diag}(1, 1, -1)$. The isometry group $SL(2, \mathbf{R})$ of the classical AdS_2 is a symmetry of this algebra, and thus is also the isometry group of the fuzzy AdS_2 .

For later use, we define the raising and lowering operators

$$X_{\pm} \equiv X^{-1} \pm iX^0, \quad (7.9)$$

which satisfy

$$[X^1, X_{\pm}] = \pm X_{\pm}, \quad [X_+, X_-] = -2X^1, \quad (7.10)$$

according to (7.7).

The radial coordinate r and the boundary time coordinate t are defined in terms of the X 's as

$$r = X^{-1} + X^1, \quad t = \frac{R}{2}(r^{-1}X^0 + X^0r^{-1}), \quad (7.11)$$

where we symmetrized the products of r^{-1} and X^0 so that t is a Hermitian operator. The metric in terms of these coordinates assumes the form (7.1). It follows that the commutation relation for r and t is

$$[r, t] = -iRl_p. \quad (7.12)$$

The following simple heuristic argument also suggests this commutation relation. Consider a closed string in AdS_2 . (Since the space is one dimensional, the closed string actually looks like an open string with twice the tension.) Take the Nambu-Goto action for a fractional string of tension $1/N$ and take the static gauge $t = p_0\tau$. It follows that the action is

$$\begin{aligned} S &= \frac{1}{2\pi N\alpha'} \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\sigma \sqrt{(p_0\dot{r}^2)} \\ &= \frac{1}{\pi N\alpha'} \int dt \int_0^{\pi} p_0 |\dot{r}| \\ &= \frac{1}{N\alpha'} \int d\tau \dot{r} \frac{\partial t}{\partial \tau}, \end{aligned} \quad (7.13)$$

where we have assumed that $\dot{r} > 0$ for $0 < \sigma < \pi$ and $\dot{r} < 0$ for $\pi < \sigma < 2\pi$. Now repeating an argument of sect.5, we conclude that there is the uncertainty relation $\delta r \delta t > Rl_p$.

7.2. Properties of Fuzzy AdS_2

One can realize the algebra (7.7) which is the same as the Lie algebra of $SL(2, \mathbf{R})$, on a unitary irreducible representation of $SL(2, \mathbf{R})$. The question is which representation is the correct choice. Since the range of X_1 goes from $-\infty$ to ∞ for AdS_2 ; when $R > 1/2$, the proper choice is the principal continuous series, Since we have $R = N > 1$ for our physical system, we should consider the principal continuous series only. A representation in this series is labeled by two parameters $j = 1/2 + is$ and α , where s, α are real numbers, and $0 \leq \alpha < 1$. The label j determines the second Casimir as

$$c_2 = \eta_{ij} X^i X^j. \quad (7.14)$$

It follows from (7.8) and $R = N$ that one should take

$$j = 1/2 + iN. \quad (7.15)$$

We set $\alpha = 0$ to focus on the case in which the reflection symmetry, $X^1 \rightarrow -X^1$ is not broken. We will denote this representation by \mathcal{D}_N and focus on this case in the following.

Functions on the fuzzy AdS_2 are functions of the X 's. They form representations of the isometry group $SL(2, \mathbf{R})$. The three generators of the isometry groups act on X as

$$[L_{ij}, X^k] = i(\delta_j^k X_i - \delta_i^k X_j), \quad (7.16)$$

where $X_i = \eta_{ij} X^j$. A very interesting property of the algebra of fuzzy AdS_2 is that the action of the generators L_{ij} is the same as the adjoint action of the X_k . That is,

$$[L_{ij}, f(X)] = \epsilon_{ijk} [X^k, f(X)] \quad (7.17)$$

for an arbitrary function $f(X)$. The operators L_{ij} act on the functions as differential operators.

The integration over the fuzzy AdS_2 is just the trace of the representation \mathcal{D}_N

$$\int f(X) \equiv c \text{Tr}(f(X)) = c \sum_{n \in \mathbb{Z}} \langle n | f(X) | n \rangle, \quad (7.18)$$

where c is a real number. This integration is invariant under $SL(2, \mathbf{R})$ transformations. In the large N limit, $c_2 \gg 1$, the trace can be calculated and its comparison with an ordinary integration on the classical AdS_2 with metric (7.1) shows that $c = N$ in the leading power

of N . The inner product of two functions, as well as the norm of a function are defined by integration over the fuzzy AdS_2 in the usual way:

$$\langle f(X)|g(X)\rangle = \int f^\dagger(X)g(X), \quad \|f(X)\|^2 = \int f(X)^\dagger f(X). \quad (7.19)$$

In view of organizing the functions into representations of $SL(2, \mathbf{R})$, in order to describe the boundary CFT dual to the bulk theory on fuzzy AdS_2 via holography, we find all functions corresponding to the lowest and highest weight states in the principal discrete series. The information about the underlying fuzzy AdS_2 , i.e., the value of N , is encoded in the precise expressions of these functions.

Denote the lowest weight state by Ψ_{jj} , or just Ψ_j . The states of higher weights Ψ_{jm} ($m > j$) in the same irreducible representation are obtained as $[X_+, [X_+, \dots [X_+, \Psi_j] \dots]]$, where X_+ appears $(m - j)$ times. With some calculation, we find the explicit expressions for the functions Ψ_j as

$$\Psi_j = \left(\frac{1}{X^1(X^1 - 1) + c_2} X_+ \right)^j. \quad (7.20)$$

In the large N limit, using $c_2 = R^2$ and the following coordinate transformation

$$X^1 = R \cot(u^+ - u^-), \quad X_\pm = \frac{R}{\sin(u^+ - u^-)} e^{\mp i(u^+ + u^-)}, \quad (7.21)$$

one finds

$$\Psi_j \rightarrow \left(\frac{e^{-iu^+} - e^{-iu^-}}{2iR} \right)^j, \quad (7.22)$$

where u^+ , u^- are the coordinates appearing in the global parametrization of AdS_2 , in agreement with the classical result [41].

Let

$$I_{jm} \equiv \text{Tr}(\Psi_{jm}^\dagger \Psi_{jm}), \quad (7.23)$$

then the normalized states are

$$\tilde{\Psi}_{jm} \equiv \frac{1}{\sqrt{cI_{jm}}} \Psi_{jm}. \quad (7.24)$$

As a normalized basis of an $SL(2, \mathbf{R})$ representation, they satisfy

$$\begin{aligned} [X_+, \tilde{\Psi}_{jm}] &= a_{j(m+1)} \tilde{\Psi}_{j(m+1)}, \\ [X_-, \tilde{\Psi}_{jm}] &= a_{jm} \tilde{\Psi}_{j(m-1)}, \end{aligned} \quad (7.25)$$

where

$$a_{jm} = \sqrt{m(m-1) - j(j-1)}. \quad (7.26)$$

For a field Φ in the bulk of the fuzzy AdS_2 , one can decompose it into the basis functions $\tilde{\Psi}_{jm}$ as

$$\Phi(X) = \sum_{jm} \phi_{jm} \tilde{\Psi}_{jm}(X), \quad (7.27)$$

where ϕ_{jm} are the creation/annihilation operators of the physical state with the wave function $\tilde{\Psi}(X)$. By holography, these states are identified with those in the boundary theory, which is a one-dimensional theory.

An interesting question for a wave function on noncommutative theory is how to define expectation values. For instance, should the expectation value of X_1 for a wave function Ψ be

$$(1) \quad \int \Psi^\dagger X^1 \Psi, \quad (2) \quad \int \Psi X^1 \Psi^\dagger, \quad (3) \quad \frac{1}{2} \int (X^1 \Psi^\dagger \Psi + \Psi \Psi^\dagger X^1)? \quad (7.28)$$

The answer is that it depends on how one measures it. If one measures the X^1 location of the wave function according to its interaction with another field Φ under control in the experiment, and if the interaction is described in the action by a term like

$$\int \Psi^\dagger \Phi \Psi, \quad (7.29)$$

then we expect that the choice (1) is correct. But if the interaction is written differently, the definition of expectation value should be modified accordingly.

7.3. Interaction in Fuzzy AdS_2

To see how the noncommutativity of the fuzzy AdS_2 incorporate physical data, presumably including the effect of string quantization on AdS_2 , we consider an interaction term in the action of the bulk theory of the form

$$S_I = \lambda \int \Phi_1^\dagger \Phi_2 \Phi_3, \quad (7.30)$$

where λ is the coupling constant for this three point interaction.

Expanding all three Φ_i 's as (7.27) in the action, one obtains the vertex

$$c\lambda \text{Tr}(\tilde{\Psi}_{j_1 m_1}^\dagger \tilde{\Psi}_{j_2 m_2} \tilde{\Psi}_{j_3 m_3}) \quad (7.31)$$

for the three states $(\phi_1)_{j_1 m_1}$, $(\phi_2)_{j_2 m_2}$ and $(\phi_3)_{j_3 m_3}$. Obviously, due to the isometry, the vertex vanishes unless $m_1 = m_2 + m_3$.

For simplicity, consider the special case where all three states participating the interaction are lowest weight states. Then the vertex (7.31) in question is $\lambda V_{m_1 m_2 m_3}$, where (for $m_1 = m_2 + m_3$)

$$V_{m_1 m_2 m_3}^2 = \frac{1}{c} \frac{I_{m_1}}{I_{m_2} I_{m_3}}. \quad (7.32)$$

After considerable calculations, we find that

$$I_m = \frac{(2m-2)!}{((m-1)!)^2} \left[\prod_{k=1}^{m-1} \frac{1}{j^2 - 1 + 4c_2} \right] I_1, \quad (7.33)$$

where

$$I_1 = \frac{\pi}{\sqrt{c_2 - 1/4}} \tanh \left(\pi \sqrt{c_2 - 1/4} \right). \quad (7.34)$$

We therefore have the large N expansion of the vertex (7.32). Using (7.15), one finds (for $m_1 = m_2 + m_3$)

$$V_{m_1 m_2 m_3}^2 = \mathcal{N}_{m_2 m_3} \frac{[\prod_{j_2=1}^{m_2-1} (1 + j_2^2/4N^2)] [\prod_{j_3=1}^{m_3-1} (1 + j_3^2/4N^2)]}{8\pi^2 N^2 [\prod_{j_1=1}^{m_1-1} (1 + j_1^2/4N^2)]} \left(1 + 2 \sum_{n=1}^{\infty} e^{-2\pi n N} \right), \quad (7.35)$$

where

$$\mathcal{N}_{m_2 m_3} = \frac{[2(m_1-1)!][(m_2-1)!]^2[(m_3-1)!]^2}{[(m_1-1)!]^2[2(m_2-1)!][2(m_3-1)!]}. \quad (7.36)$$

With the possibility of corrections to (7.15) of order $(1/N)^0 \sim 1$, the $1/N$ expansion of the vertex is of the form

$$V_{m_1 m_2 m_3} \sim \frac{K}{N} \left(1 + \sum_{n=1}^{\infty} \frac{a_n}{N^{2n}} + \sum_{n=1}^{\infty} e^{-2\pi n N} \sum_{k=0}^{\infty} \frac{b_{nk}}{N^{2k}} \right). \quad (7.37)$$

This expression is reminiscent of a correlation function in the case of type IIB strings on $AdS_5 \times S^5$ [42]. It calls for an analogous interpretation.

The $1/N^2$ expansion in (7.37) suggests that the coupling constant in AdS_2 is of order $1/N^2$. This is indeed the case. The 11 dimensional Newton constant is just 1 in Planck units. Compactifying on $S^2 \times T^7$ of size $4\pi N^2$ results in a dimensionless effective Newton constant of order $1/N^2$ in AdS_2 .

The overall factor of $1/N$ in (7.37) is what one expects for a three-point correlation function, since for a large N theory, the string coupling constant is proportional to $1/N$,

it appears in the three-point coupling if the connected two-point function is normalized to one.

Finally, we identify the terms in (7.37) proportional to $\exp(-2\pi nN)$ as contributions from instantons. This implies that the action of a single instanton equals $2\pi N$. We have just argued that the string coupling constant g_s is of order $1/N$. If the instanton action is $2\pi/g_s$, it is precisely $2\pi N$ as we wish.

Similarly, we can consider n -point interaction vertex in the bulk theory on AdS_2 :

$$S_n = \lambda \int \Phi_1^\dagger \Phi_2 \cdots \Phi_n. \quad (7.38)$$

The leading dependence of the vertex will be $1/N^{n-2}$, which is exactly what it should be for an n -point function in string theory with coupling constant $g_s \sim 1/N$.

We conjecture that for M theory compactified on $AdS_2 \times S^2$, the perturbative as well as nonperturbative effects of string quantization are encoded in the noncommutativity of the fuzzy $AdS_2 \times S^2$, in the sense that the low energy effective theory is most economically written as a field theory on this noncommutative space.

7.4. A Shock-Wave Argument

Although we already argued for the spacetime uncertainty in the fuzzy AdS_2 from the general stringy uncertainty, it should be interesting to compare the way 't Hooft introduces spacetime noncommutativity in his S-matrix ansatz. The method is similar to that in sect.4.

Use the global coordinates. The metric induced by a left-moving shock-wave assumes the form

$$ds^2 = -e^{2\phi} du^+ du^- + h(du^+)^2, \quad (7.39)$$

where

$$e^{2\phi} = \frac{4R^2}{\sin^2(u^+ - u^-)}.$$

The scalar curvature is perturbed by a term

$$\frac{1}{2} e^{-2\phi} \partial_- (e^{-2\phi} \partial_- h), \quad (7.40)$$

and the Einstein equation with a constant negative curvature is solved provided

$$h = \cot(u^+ - u^-) g(u^+) + f(u^+). \quad (7.41)$$

In the full 4 dimensions, we expect another Einstein equation of the form $G_{++} = 8\pi G_4 T_{++}$, where G_4 is the 4D Newton constant. Now $G_{++} \sim h\mathcal{R}$, \mathcal{R} is the scalar curvature. For a stress tensor T_{++} proportional to $\delta(u^+ - x^+)$, the only solution is $g(u^+) = 0$ and

$$h \sim l_p^2 \delta(u^+ - x^+). \quad (7.42)$$

The proportionality constant is determined by how the stress tensor is normalized. For a S-wave shock-wave smeared over S^2 , it is p_+/R^2 . However, the dipole mechanism of [15] seems to indicate that a shock-wave must be localized on a strip on S^2 whose area is proportional to Nl_p^2 . If true, we expect

$$\int du^+ T_{++} \sim \frac{p_+}{Rl_p}, \quad (7.43)$$

and this leads to

$$h \sim Rl_p p_+ \delta(u^+ - x^+). \quad (7.44)$$

Now the shift on u^- induced on a right-moving particle by the shock-wave is

$$\Delta u^- \sim \frac{p_+}{N} \sin^2(x^+ - u^-) \quad (7.45)$$

as can be computed using (7.39). This shift suggests a commutator

$$[u^+, u^-] \sim \frac{i}{N} \sin^2(u^+ - u^-), \quad (7.46)$$

the one that is compatible with our fuzzy AdS_2 model.

8. Conclusions

We have amassed a few scattered aspects of the fuzzy spacetime ranging from some elementary study of quantum horizon of black holes by 't Hooft, to stringy spacetime uncertainty, to fuzzy anti-de Sitter space. It can not be over-emphasized that further and deepened study of all these aspects is one of most urgent tasks in string/M theory. Here, instead of pointing out problems already well-defined without much a quandary, we ask a few questions.

1. In quantum mechanics, observables are operators generally noncommuting. However, one rarely associates an observable to time. In this vein, we ask: What is the most fundamental meaning of noncommutativity of space and time? It appears that this notion

challenges our usual understanding of both general relativity and quantum mechanics. On the one hand, now an event is not a well-defined concept, thus challenging the very foundation of general relativity. On the other hand, if time is not a usual number, one needs to extend quantum mechanics either in the Heisenberg form or in the Schrödinger form, since in both one equates the time derivative with the Hamiltonian operator.

2. Clearly spacetime uncertainty is a generic feature of string/M theory, do we need to formulate string/M theory in a manifestly noncommutative fashion?
3. What is the relation between noncommutative horizon of black holes and stringy noncommutativity? The latter sounds more microscopic, while the former is more effective but presumably includes nonperturbative information.
4. Can one derive the entropy and other physical quantities (such as modified Hawking radiation) of black holes from noncommutativity?
5. Does noncommutativity in string theory violate causality by an arbitrary amount when the energy is increased, presumably at a nonperturbative level?
6. How to derive holography from noncommutativity? Or does there exist a bulk formulation which incorporates noncommutativity explicitly?

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